

High-Rate Short-Block LDPC Codes for Iterative Decoding with Applications to High-Density Magnetic Recording Channels

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Abstract—This paper investigates the *Triangle Single Parity Check* (T/SPC) code, a novel class of high-rate low-complexity LDPC codes. T/SPC is a regular, soft decodable, linear-time encodable/decodable code. Compared to previous high-rate and low-complexity LDPC codes, such as the well-known *Turbo Product Code / Single Parity Check* (TPC/SPC), T/SPC provides higher code rates, shorter code words, and lower complexity. This makes T/SPC very attractive for practical implementation on integrated circuits.

In addition, we analyze the performance of iterative decoders based on a soft-input soft-output (SISO) equalizer using T/SPC over high-density perpendicular magnetic recording channels. Computer simulations show that the proposed scheme is able to achieve a gain of up to 0.3 dB over TPC/SPC codes with a significant reduction of implementation complexity.

I. INTRODUCTION

Iterative decoders based on soft-input soft-output equalizers with powerful error correction codes (ECC) -such as low density parity check (LDPC)- are considered in the literature to be suitable for coping with intersymbol interference and noise in high-speed transmission systems [1], [2], [3]. The potential of this architecture is high, but it requires additional improvements in order to be applicable to the next generation of magnetic recording systems. There are mainly two reasons for this: the high complexity of implementing high-speed iterative receivers in integrated circuits (e.g., 4 Gb/s or higher), and the potential *error floor* problem of fully iterative decoding solutions [4], [5], [6]. The latter problem is exacerbated by the difficulty of evaluating performance at very low bit error rates (BERs).

One interesting alternative that offers a good tradeoff between complexity and performance is the combination of (i) an iterative scheme based on a SISO equalizer with an *inner* high-rate low-complexity LDPC code and (ii) a powerful *outer* code (such as RS, BCH, or Goppa), as shown in Fig. 1 [7]. In this scheme, the inner LDPC code should achieve very high rates with low complexity as well as provide good error statistics to the outer ECC. It has been shown that the *Turbo Product Code / Single Parity Check* (TPC/SPC) is a suitable candidate to be used as inner code [8], [9]. The main features

of the TPC/SPC codes are: minimum distance $d_H = 4$, no length-4 cycles, linear-time encodable and decodable (i.e., low complexity implementation), and high code rate for relatively short codewords [8], [9], [10]. In magnetic recording systems, where the sector size is fixed, the use of LDPC with shorter code words provides benefits in an iterative architecture. This is because the combination of interleaving with *various* code words per sector gives rise to interleaving gain. Therefore, the design of high rate short block LDPC codes holds great interest for iterative receivers in high-density magnetic recording systems.

In this paper we investigate the *Triangle Single Parity Check* (T/SPC) code, a novel LDPC code that can be derived from combinatorial design criteria [9], [11], [12], [13]. T/SPC is suitable as inner code in the iterative decoding scheme shown in Fig. 1. Compared to TPC/SPC, the proposed T/SPC exhibits:

- half code length for a given code rate;
- half parity check nodes for a given code rate;
- higher code rates for a given sector length;
- lower minimum distance ($d_H = 3$).

Note that T/SPC is able to provide a significant reduction of complexity but this comes at the expense of a lower minimum distance when compared to TPC/SPC. However, as it will be shown later, combining a lower code length together with an interleaver allows T/SPC to achieve a significant interleaving gain. Furthermore, numerical results show that T/SPC is able to provide a 0.3 dB gain over TPC/SPC in magnetic recording systems.

The rest of the paper is organized as follows. The system model is presented in Section II. T/SPC code is analyzed in Section III. Section IV evaluates the performance of T/SPC and TPC/SPC. Finally, conclusions are drawn in Section V.

II. SYSTEM MODEL

The system model under study is shown in Fig. 1. Information bits are first encoded with an ECC such as RS or BCH code (i.e., the outer code), and the output of the ECC is then encoded with an LDPC code (i.e., the inner code). LDPC

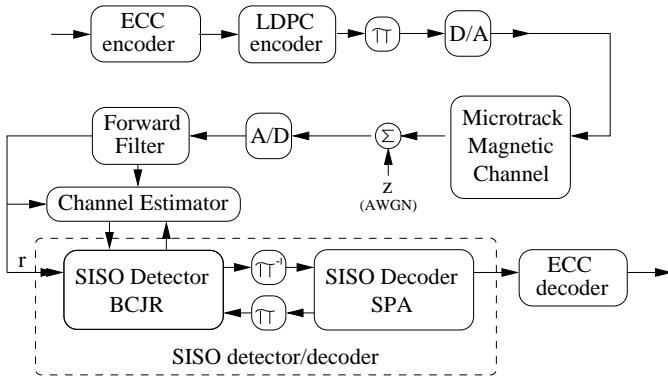


Figure 1. System Model

codes considered here are T/SPC and TPC/SPC. Codewords at the LDPC output are interleaved using a random-design interleaver. At the receiver side, samples are first processed by an adaptive linear feed forward filter (FFF) in order to adapt the channel response to the desired target response. Finally, the FFF output is used by the iterative detector to estimate the transmitted bit sequence. The SISO detector is implemented using the Max-Log-MAP approximation of the BCJR algorithm [14]. The SISO decoder runs with two iterations of the min-sum approximation of Sum-Product-Algorithm (SPA) [10].

A. Channel Model

The microtrack model [15] is used for the magnetic recording channel with signal dependent transition noise. The received signal is given by

$$s(t) = \frac{1}{N_t} \sum_k \sum_{n=1}^{N_t} b_k \cdot h(t - T_k - \tau_{k,n}) + z(t) \quad (1)$$

where:

- $h(t)$ is the transition response of a simple microtrack.
 $h(t) = V \cdot \text{erf} \left(\frac{2\sqrt{\ln(2)} \cdot t}{PW_{50}} \right)$, where V is the transition amplitude, PW_{50} is defined as the width of the derivative of $h(t)$ at half its peak amplitude.
- $b_k \in \{-1, +1\}$ represents the transitions direction.
- T_k is the ideal time in which the k th transition takes place.
- $\tau_{k,n}$ is the random shift of the k th transition on the n th microtrack.
- N_t is the total number of microtracks ($N_t = 2$ is used in this paper).
- $z(t)$ is electronic noise.

The density is defined as $D = \frac{PW_{50}}{T}$, where T is the bit period. The transition jitter noise $\tau_{k,n}$ is modeled as an i.i.d. white Gaussian random variable. The signal-to-noise ratio (SNR) is defined as $SNR = \frac{1}{\sigma_z^2 + \sigma_j^2}$ ($V = 1$ is assumed), where σ_z^2 is the power of the electronic noise samples, and σ_j^2 the power of the media noise component [9]. Media noise is defined by $s_j(t) = s_{sj}(t) - s_s(t)$, where $s_{sj}(t)$ is the

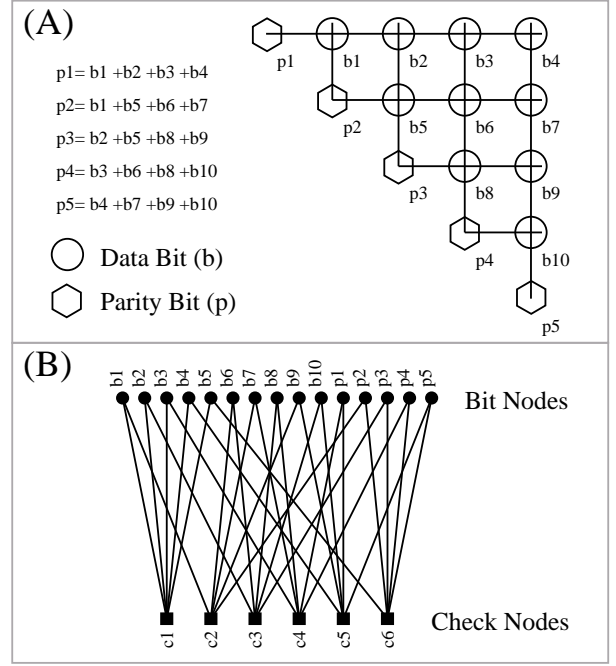


Figure 2. A) Parity bits structure of a T/SPC(5) code B) Tanner graph of a T/SPC(5) code.

microtrack channel output and $s_s(t)$ the jitter-free microtrack channel output.

III. THE TRIANGLE/SPC CODE

T/SPC code is a subclass of LDPC codes, obtained from the combination of single parity check codes. A special case is the two-dimensional T/SPC, denoted as 2D-T/SPC, where the data and parity bits of the codeword are arranged in a 2-dimensional triangular array as shown in Fig. 2-A. Its Tanner graph is depicted in Fig. 2-B. Note that the 2D-T/SPC is a systematic code with the single parity bits as redundancy. This code can be completely characterized by the number of nodes in the edges of the triangle, N . The parameters of a 2D-T/SPC(N) code are:

Minimum dist.:	$d_{2,N}$	$= 3$
Code length:	$n_{2,N}$	$= N(N + 1)/2$
Code dimension:	$k_{2,N}$	$= N(N - 1)/2$
Code rate:	$R_{2,N}$	$= (N - 1)/(N + 1)$
Check equations:	$c_{2,N}$	$= N + 1$

A 2D-T/SPC code can be extended to an M -dimensional T/SPC (MD -T/SPC) code by building an M -dimensional triangular array. The code parameters of the resulting MD -T/SPC(N) are:

Minimum dist.:	$d_{M,N}$	$= M + 1$
Code length:	$n_{M,N}$	$= \sum_{i=1}^N n_{M-1,i}$
Code dimension:	$k_{M,N}$	$= \sum_{i=1}^N k_{M-1,i}$
Code rate:	$R_{M,N}$	$= k_{M,N}/n_{M,N}$
Check equations:	$c_{M,N}$	$= \sum_{i=1}^N c_{M-1,i}$

We focus on the 2D-T/SPC code that achieves a higher code

rate to code-length ratio than the 2D-TPC/SPC. We first show that the 2D-T/SPC code, as the 2D-TPC/SPC, is a special case of *Combinatorial Design Codes*, and later the lower complexity of the T/SPC compared with the TPC/SPC.

A. T/SPC and Combinatorial Design Codes

The T/SPC can be analyzed using *combinatorial design*. A combinatorial design is an arrangement of a set of m points into n subsets, called blocks, which satisfy certain regularity constraints [9], [11], [12], [13]. The *covalency* λ_{v_1, v_2} of two points v_1 and v_2 is the number of blocks that contain both of them. If λ_{v_1, v_2} is the same for all pairs of points, the design is said to be *balanced*. The number of points contained in each block and the number of blocks each point is incident with, are denoted by γ and ρ , respectively. If γ and ρ are the same for each block and point, respectively, the design is said to be *regular*. A regular and balanced design is denoted as a $(m, n, \rho, \gamma, \lambda)$ -design. The *incidence matrix* M of the combinatorial design has dimension $n \times m$, where $M_{i,j} = 1$ if the point v_j is incident with the block B_i and $M_{i,j} = 0$ otherwise.

The transpose of the incidence matrix may be used as the parity check matrix H of an LDPC code. In this construction of an LDPC code, a point in a combinatorial design corresponds to a *check node* or a row in H , and a block corresponds to a *bit node* or a column in H . The row and column weights of H are ρ and γ respectively. A covalency $\lambda < 2$ guarantees the absence of length-4 cycles in the Tanner graph. For example, the 2D-TPC/SPC($N, N-1$)² can be constructed from a $(2\rho, \rho^2, \rho, 2, \{0, 1\})$ -design [13] where $\rho = N$.

One of the most interesting classes of combinatorial design are the *Steiner systems* [11], [16]. A (m, γ, t) -Steiner-system is a set \mathcal{M} of m points, and a collection \mathcal{N} of subsets of \mathcal{M} of size γ , called blocks, such that any subset of t points of \mathcal{M} is in exactly one of the blocks. The size of a Steiner system, n , is defined as the number of blocks:

$$n = |\mathcal{N}| = \binom{m}{t} / \binom{\gamma}{t} \quad (2)$$

In [11], [12], [16] the construction and performance of LDPC codes based on a $(m, 3, t)$ -Steiner-system (called Steiner triple systems), and a particular case called Kirkman system, are analyzed. We will focus on the $(N-1, 2, 2)$ -Steiner-system which produces a $(m, n, \rho, \gamma, \lambda)$ -design with parameters: $n = N(N+1)/2$, $m = N+1$, $\rho = N$, $\gamma = 2$, $\lambda = 1$. The code obtained from the incidence matrix of this design is the 2D-T/SPC(N).

B. Complexity of T/SPC and TPC/SPC

The code rate of a 2D-TPC/SPC($N, N-1$)² is given by

$$\begin{aligned} R_{TPC/SPC} &= \left(\frac{N-1}{N} \right)^2 \\ &= \frac{N-1}{N+1} - \frac{1}{N(N+1)} + \frac{1}{N^2(N+1)} \\ &\approx \frac{N-1}{N+1} = R_{T/SPC} \quad \text{for } N \gg 1 \end{aligned} \quad (3)$$

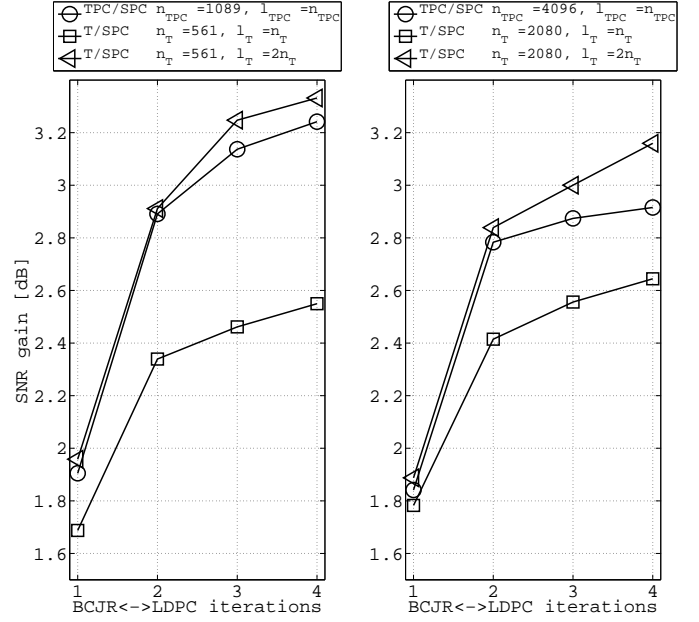


Figure 3. SNR gain vs. number of BCJR ↔ LDPC iterations at BER = 10⁻⁴. Code rates 0.94 (left) and 0.969 (right).

Therefore, a simple complexity comparison between 2D-T/SPC and 2D-TPC/SPC can be done considering the same parameter N for both codes.

Given that complexity is mainly on the decoder, we will focus on the SPA-based decoder. T/SPC requires approximately the same bit-nodes and check-nodes computations as TPC/SPC. In general, the memory requirement and the interconnection complexity are proportional to the number of edges (N_e) of the code factor graph. Considering that the number of edges is $N_e = N(N+1)$ for T/SPC and $N_e = 2N^2$ for TPC/SPC, we conclude that T/SPC is able to provide a significant reduction of memory and interconnection complexity ($\sim 1/2$ for $N \gg 1$).

IV. NUMERICAL RESULTS

The performance of T/SPC and TPC/SPC is analyzed by using computer simulations of the system described in Section II. Code rates of 0.94 and 0.969 are analyzed. We consider a perpendicular channel of density $D = 3$ and 80% / 20% jitter/electronic noise power (i.e., $\sigma_j^2/\sigma_z^2 = 0.8/0.2$). A generalized partial response target with 4 taps (GPR4) is utilized. The LDPC decoder performs two SPA iterations. Random-design interleavers are used.

Let l_T and l_{TPC} be the interleaver lengths used on the T/SPC and TPC/SPC codes respectively. Also, let n_T and n_{TPC} be the code-word lengths for the T/SPC and TPC/SPC codes respectively. For TPC/SPC we use $l_{TPC} = n_{TPC}$. On the other hand, we analyze T/SPC with two interleaver lengths: $l_T = n_T$ and $l_T = 2n_T \approx l_{TPC}$. Figure 3 shows the SNR gain vs. number of BCJR ↔ LDPC iterations at BER = 10⁻⁴. The performance achieved by an uncoded BCJR equalizer is

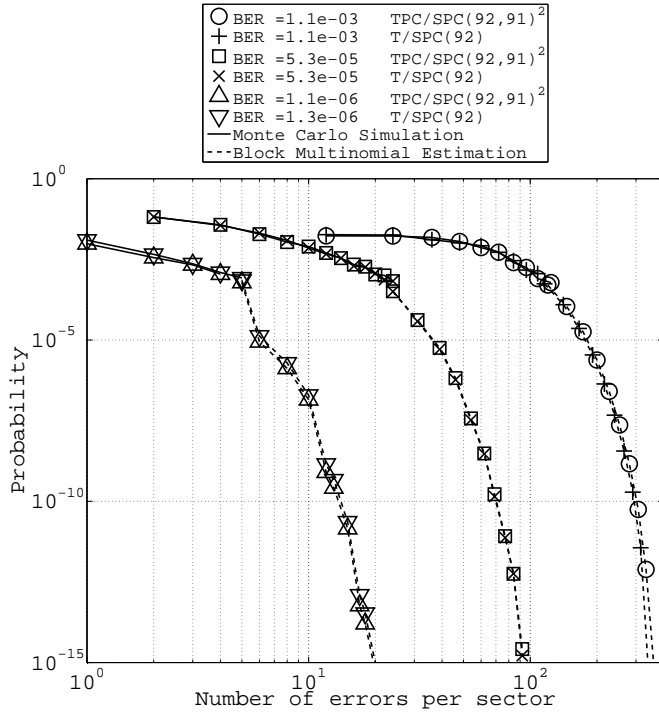


Figure 4. Probability of the number of errors per sector over a PR[1 3 3 1] with 100% electronic noise. Four TPC/SPC(92,91)² codes and eight T/SPC(92) codes per sector.

taken as reference. The resulting performance of T/SPC for $l_T = n_T$ is worse than the one of TPC/SPC, mainly because of the T/SPC lower minimum distance. However, for $l_T = 2n_T \approx l_{TPC}$ the interleaver gain compensates the degradation caused by its lower minimum distance. Furthermore, in some cases note that T/SPC is able to provide a gain of around 0.27 dB over TPC/SPC owing to the interleaving gain. Note that this gain is achieved with lower complexity.

In the iterative-based receiver depicted in Fig. 1, the statistics of the bit errors at the output of the inner code strongly affect the performance of the outer code. Next we investigate the distribution of bit errors at the output of the T/SPC and TPC/SPC with code rate 0.978 and two BCJR \leftrightarrow LDPC iterations. Sector size is 4KB and the interleaving length is 1KB. Figure 4 shows the probability of the number of errors per sector for a partial response channel PR[1 3 3 1] with 100% electronic Gaussian noise. Solid lines correspond to values derived from simulations, while dashed lines represent estimates calculated by using the block-multinomial model [17]. From Fig. 4 we note that, at a given BER, the statistics of bit errors at the output of T/SPC are practically the same as those observed with TPC/SPC. Therefore, we infer that the behavior of T/SPC and TPC/SPC in an iterative-based receiver as shown in Fig. 1, should be similar. This topic, and other related issues, will be addressed in a future work.

V. CONCLUSIONS

We have proposed and investigated T/SPC, a novel high-rate LDPC code. T/SPC provides higher code rates, shorter code words, and lower complexity than TPC/SPC. Computer simulations of T/SPC on high-density perpendicular magnetic recording channels have shown that T/SPC is able to achieve a significant interleaving gain, outperforming TPC/SPC by almost 0.3 dB. Our results suggest that the design of LDPC with high-rate, low complexity and short code word is a promising research area for next generation developments of magnetic recording devices.

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